SOLUTIONS to TEST 2

Semester 2 2005

QUESTION 1

(a) State the steady-state error of a control system and procedures to find the steady state error. Given the following second-order system

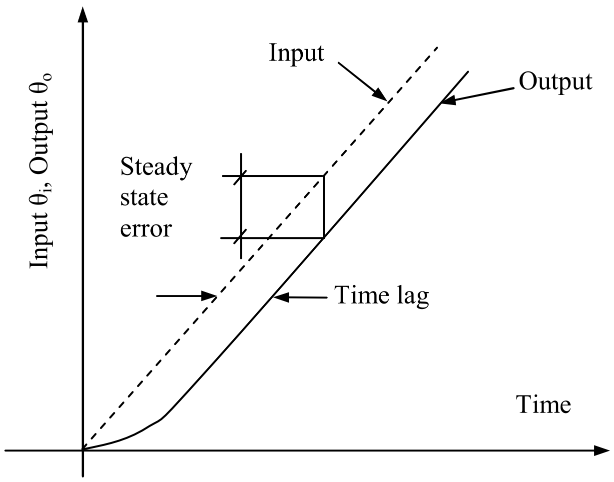
$$\ddot{y} + 2\xi\omega_n \dot{y} + \omega_n^2 y = ku$$

where k is constant, ξ is the damping ratio, ω_n is the undamped natural frequency and u and y are input and output, respectively. Find the steady state error if a unit-step test input signal is applied.

[4 marks]

SOLUTION

1. Definition of SSE: The steady state error of a control system is defined as the difference between the output signal (process variable) and the reference signal (input) under the steady state condition. The concept of the SSE is illustrated in the following figure.



(You may use a block diagram of the closed-loop control system to define the steady state error)

2. Three Steps to Calculate the Steady State Error:

Step 1: Find total feedback transfer function:

$$\text{F.B.T.F} = \frac{Y(s)}{R(s)}$$

Note that the total feedback transfer function is the ratio of the output (process variable) to the reference input (set point).

Step 2: Apply an input signal and find the Laplace transform of error by

$$E(s) = R(s)(1 - \text{F.B.T.F})$$

Step 3: Find the steady state error by

$$\lim_{s \to 0}[sR(s)(1 - \text{F.B.T.F})]$$

3. Application to find SSE:

3.1. Forward path transfer function: $G(s) = \dfrac{K}{s^2 + 2\omega_n \xi s + \omega_n^2)}$

3.2 F.B.T.F = $\dfrac{G}{1+G}$

3.3 $E(s) = R(s)(1 - F.B.T.F.) = R\left[1 - \frac{G}{1+G}\right]$

3.4 $R(s) = \frac{1}{s}$

3.5 $\lim_{s\to 0}[sR(s)(1 - F.B.T.F)] = \lim_{s\to 0}\left[s\frac{1}{s}\left(1 - \frac{G}{1+G}\right)\right] = \lim_{s\to 0}\left(\frac{1}{1+G}\right) =$

$$\lim_{s\to 0}\left(\frac{1}{1+\frac{K}{s^2+2\xi\omega_m s+\omega_n^2}}\right) = \lim_{s\to 0}\left(\frac{s^2+2\xi\omega_m s+\omega_n^2}{s^2+2\xi\omega_m s+\omega_n^2+K}\right) = \frac{\omega_n^2}{\omega_n^2+K}$$

(b) Find poles and zeros of the following transfer functions and draw them in s-plane (complex plane):

i) $H(s) = \frac{(s^2+3s+2)K}{(s^2+3s+5)(s^2+7s+12)}$ (K is a constant)

ii) $H(s) = \frac{1+3s}{3+7s+s^2}$

iii) $H(s) = \frac{(2s+1)K}{s^2+3s+9}$ (K is a constant)

iv) $H(s) = \frac{(3s^2+2s+1)K}{(s^2+5s+9)(2s^2+3s+7)}$ (K is a constant)

[4 marks]

SOLUTION

(i) $H(s) = \frac{(s^2+3s+2)K}{(s^2+3s+5)(s^2+7s+12)}$ (K is a constant)

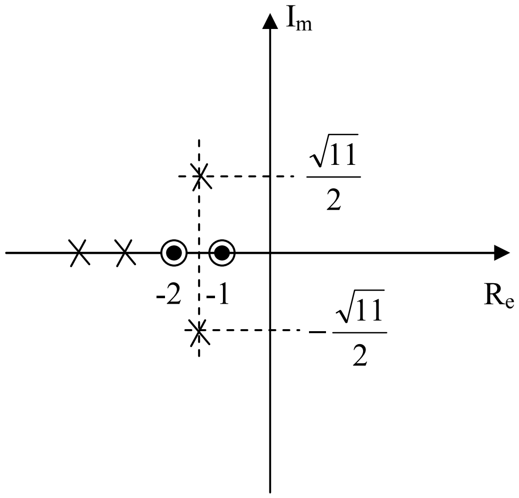
zeros: $(s^2+3s+2) = 0$

$z_{1,2} = \frac{-3 \pm \sqrt{9-4\times 2}}{2} = -1, -2$

poles: $(s^2+3s+5)(s^2+7s+12) = 0$ or $(s^2+3s+5) = 0$ and $(s^2+7s+12) = 0$

$p_{1,2} = \frac{-3 \pm \sqrt{9-4\times 5}}{2} = \frac{-3 \pm j\sqrt{11}}{2}$

$p_{3,4} = \frac{-7 \pm \sqrt{49-4\times 12}}{2} = \frac{-7 \pm \sqrt{1}}{2} = -3, -4$



(ii) $H(s) = \dfrac{1+3s}{3+7s+s^2}$

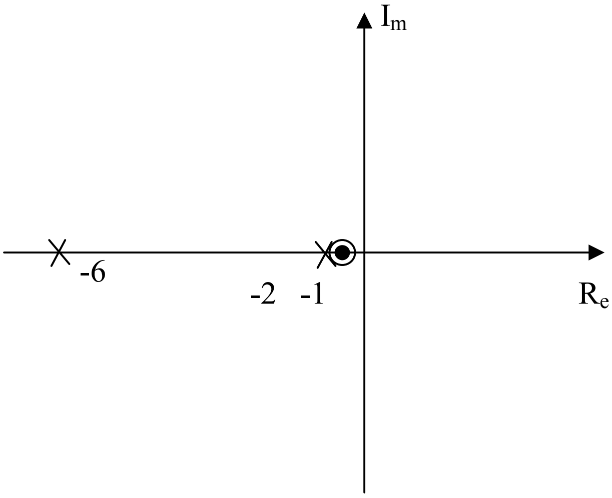
zeros: $1+3s=0$

$s = -\dfrac{1}{3}$

poles: $3+7s+s^2 = 0$

$p_{1,2} = \dfrac{-7 \pm \sqrt{49-4\times 3}}{2} = \dfrac{-7 \pm \sqrt{49-4\times 3}}{2}$

$= -0.45, -6.54$



(iii) $H(s) = \dfrac{(2s+1)K}{s^2+3s+9}$ (K is a constant)

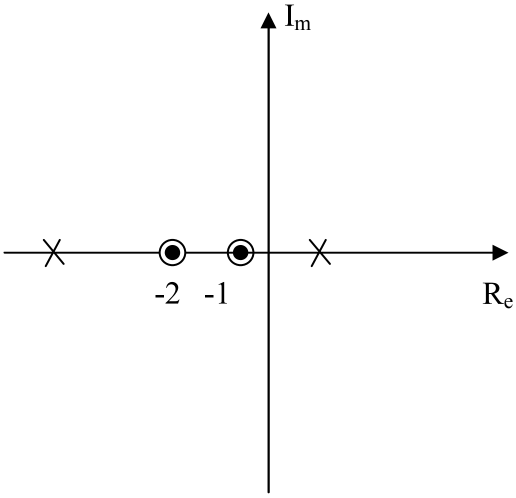
zeros: $2s+1=0$

$z = -\dfrac{1}{2}$

poles: $s^2+3s+9=0$

$p_{1,2} = \dfrac{-3 \pm \sqrt{9-4\times 9}}{2}$

$= 1.09, -4.09$



(iv) $H(s) = \dfrac{(3s^2+2s+1)K}{(s^2+5s+9)(2s^2+3s+7)}$ (K is a constant)

zeros: $3s^2+2s+1=0$

$z_{1,2} = \dfrac{-2 \pm \sqrt{4-4\times 3\times 1}}{2\times 3} = \dfrac{-2 \pm j\sqrt{8}}{6}$

$= -0.33+j0.46, -0.33-j0.46$

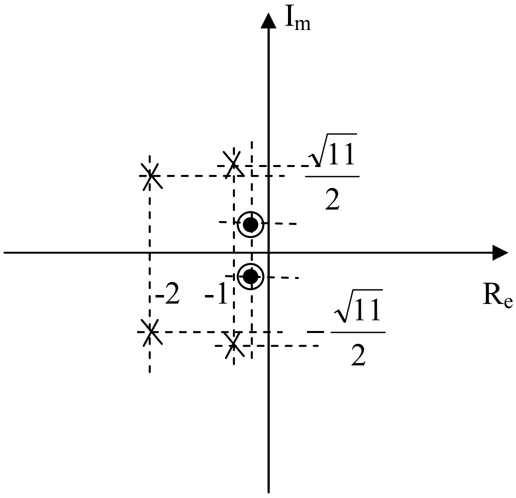
poles: $(s^2+5s+9)(2s^2+3s+7)=0$

$p_{1,2} = \dfrac{-5 \pm \sqrt{25-4\times 9}}{2} = \dfrac{-5 \pm j\sqrt{11}}{2}$

$= -2.5+j1.65, -2.5-j1.65$

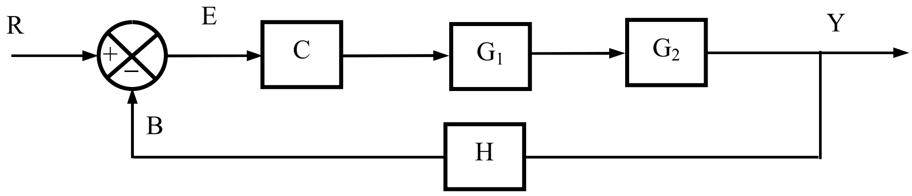
$p_{3,4} = \dfrac{-3 \pm \sqrt{9-4\times 2\times 7}}{2\times 2} = \dfrac{-3 \pm j\sqrt{47}}{4}$

$= -0.75+j1.7; -0.75-j1.7$



QUESTION 2

1. Define the block diagram to represent a dynamic system. Given the following block diagram for a feedback control system,

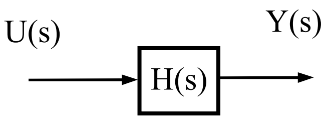


write the forward transfer function, the open-loop transfer function and the feedback (closed-loop) transfer function. Note that R is the set-point variable, E is the error, and Y is the process variable (output) and B is the feedback signal.

[4 marks]

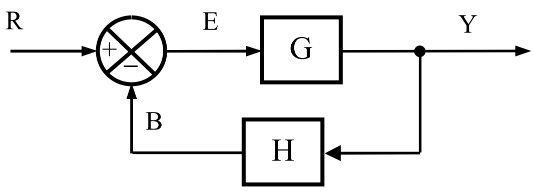
SOLUTION

Definition of block diagram: A block used to represent a dynamic system is a rectangle that contains a transfer function, an input and an output as illustrated in the following figure:

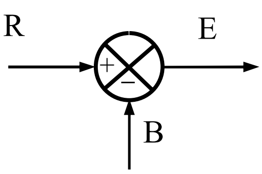


The relationship between the output and input is $H(s) = \dfrac{Y(s)}{U(s)}$ or $Y(s) = H(s)U(s)$

A combination of blocks and connections between them creates a block diagram. The following figure shows a closed-loop block diagram with a comparison element:



In a block diagram the comparison element as shown in the following figure is used to represent the sum function in which E = R – B (the difference between input signal R and the measured output signal or process variable B, also called "feedback signal").



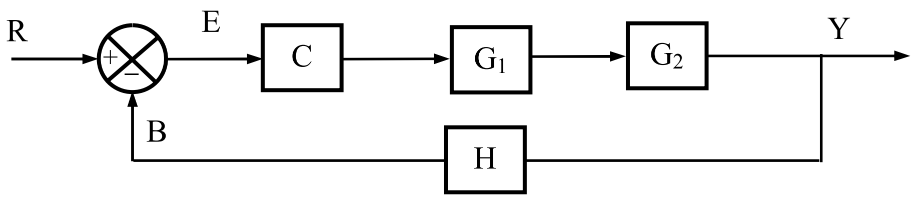
From the block diagram of a closed-loop control system as shown above:

Forward transfer function = $\dfrac{Y}{E} = G$

Open-loop transfer function = $\dfrac{B}{E} = GH$

Closed-loop (or feedback) transfer function = $\dfrac{Y}{R} = \dfrac{G}{1 + GH}$

Application: Applying the above definitions to the given block diagram we have:

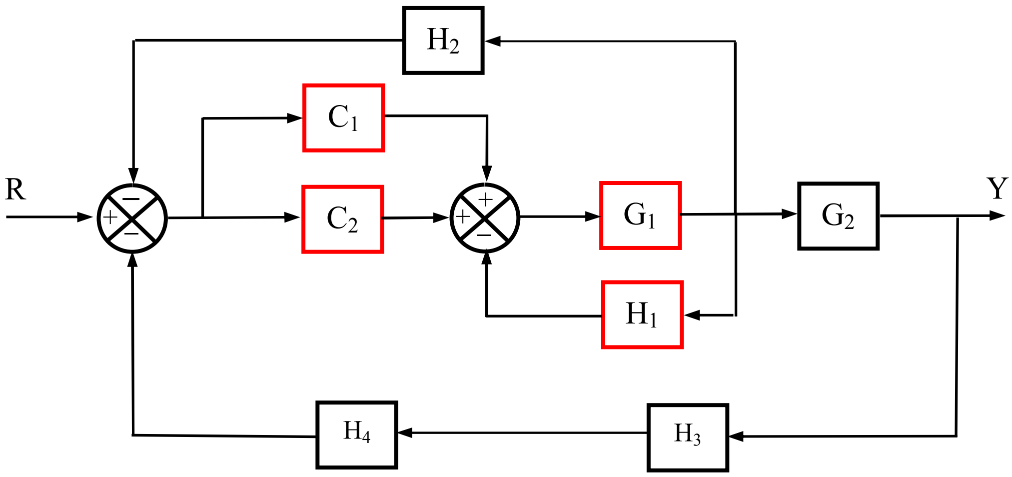


Forward transfer function = $\frac{Y}{E} = CG_1G_2$

Open-loop transfer function = $\frac{B}{E} = CG_1G_2H$

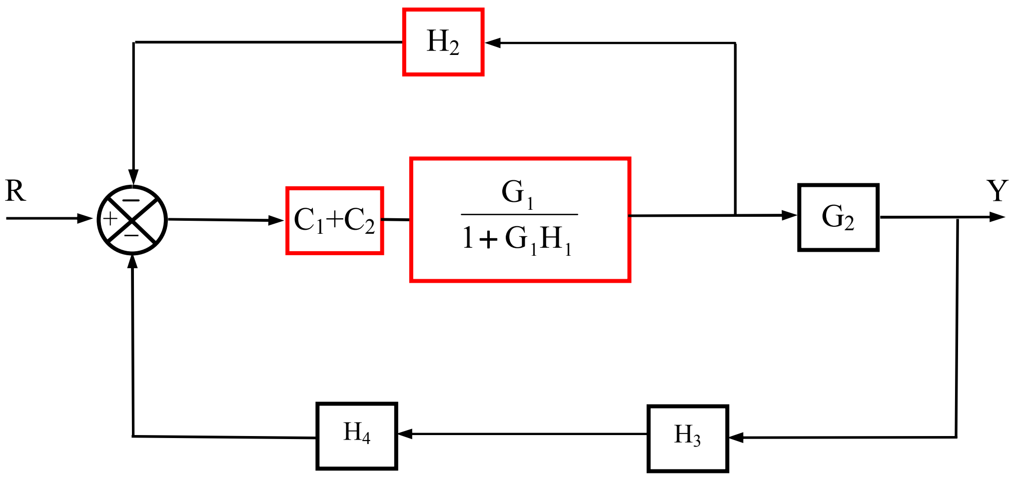
Closed-loop (or feedback) transfer function = $\frac{Y}{R} = \frac{CG_1G_2}{1 + CG_1G_2H}$

2. Find a reduced block diagram that is equivalent to the following block diagram

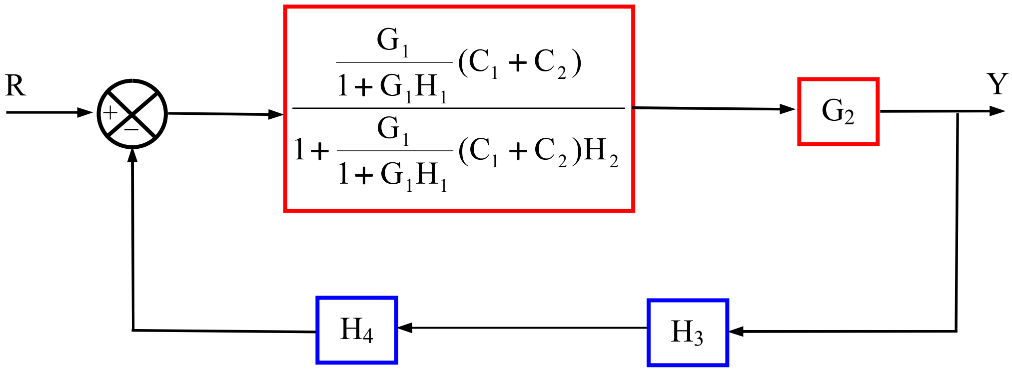


[4 marks]

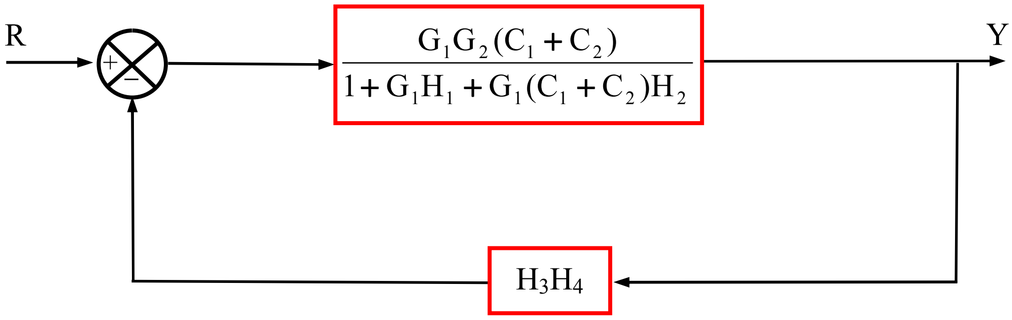
SOLUTION
(To solve this problem, just apply the block diagram algebra)
Step 1:



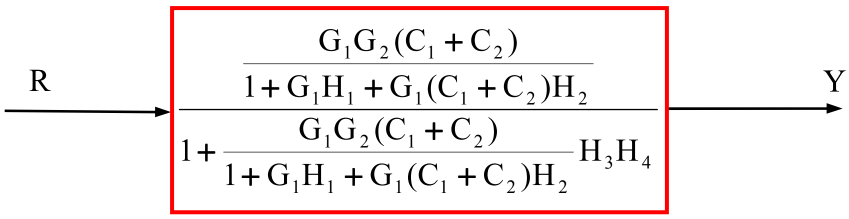
Step 2



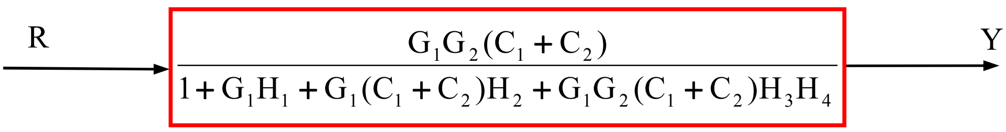
Step 3:



Step 4:



Final block diagram:



QUESTION 3

(a) Describe one final control element (actuator). You may use a block diagram, formulae and/or a schematic diagram to illustrate your answers.

[4 marks]

SOLUTION

You may choose any final control element (actuator) you have known the best. The answer may include the following:

1. Structure
2. Principle of operation
3. Features
4. Application

(*See Module 10 Control Components for one of the final control elements*)

(b) A closed-loop control system for control of remote position consists of a potentiometer typed comparison element, a PI controller with control gains of K_P and K_I, an amplifier with a gain of K_G, a motor system that has transfer function:

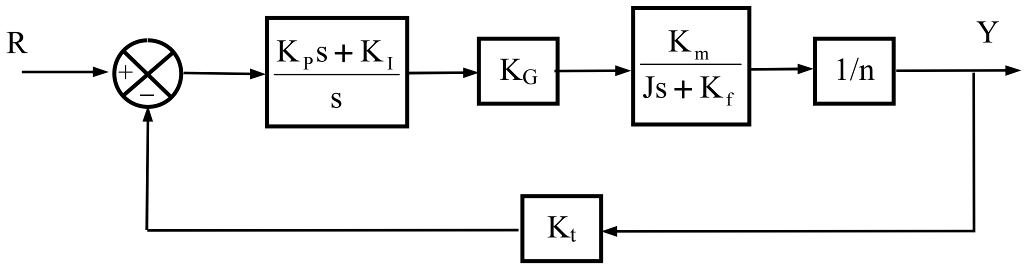
$$\frac{K_m}{Js + K_f} \text{ (where } K_m\text{, J and } K_f \text{ are constant)}$$

and a reduction gearbox that has gear ratio of 1/n. The output (process variable) of the system is measured by a tachometer that has the sensitivity of K_t. Draw a block diagram and write the total feedback transfer function for the control system.

[4 marks]

SOLUTION

1. Block diagram:



2. Feedback transfer function: Based on the above block diagram, the total feedback transfer function is:

$$\text{F.B.T.F.} = \frac{\frac{K_P s + K_I}{s} K_G \frac{K_m}{Js + K_f} \frac{1}{n}}{1 + \frac{K_P s + K_I}{s} K_G \frac{K_m}{Js + K_f} \frac{1}{n} K_t} = \frac{(K_P s + K_I) K_G K_m}{n(Js + K_f)s + (K_P s + K_I) K_G K_m K_t}$$

QUESTION 4

A computer-based liquid level control system is shown in the following figure. The tank has volume of V (m^3), cross-sectional area of A (m^2) and the liquid in the tank has constant density ρ kg/m^3.

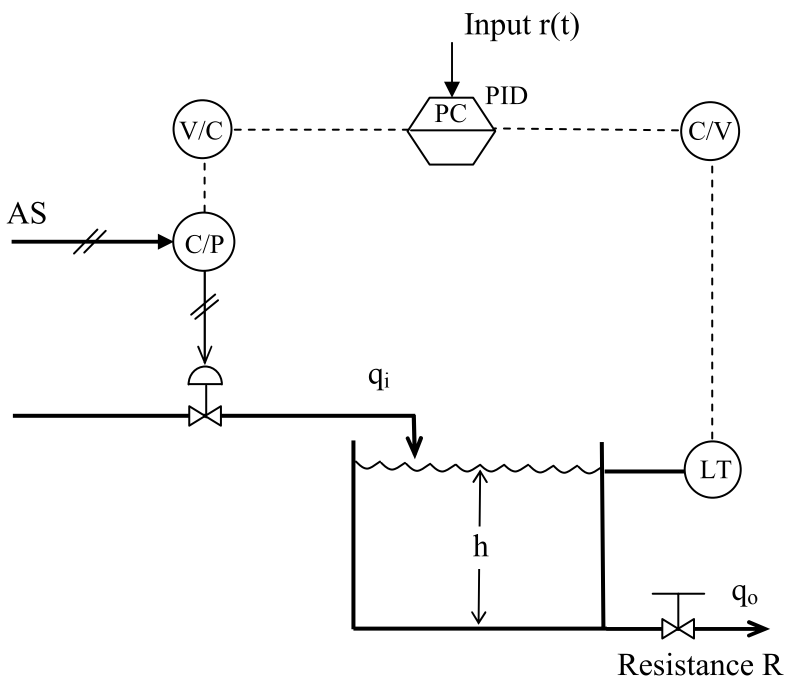


Figure (level control system) for Question 4

Assumptions for the control system are as follows:

1. q_i and q_o are inlet flow rate and outlet flow rate (m^3/sec), h is the level of the liquid in the tank (m) and AS stands for air supply.
2. LT is a level transmitter with gain of K_m
3. C/V is a current-to-voltage converter with gain of K_{CV}
4. PC is the computer in which a PID controller for the liquid system is implemented. The control gains of the PID controllers are K_P, K_I, and K_D
5. V/C is a voltage-to-current converter with gain of K_{VC}
6. C/P is a current-to-pressure converter with gain of K_{CP}
7. The control valve has a gain of K_V
8. The set-point signal r(t) (m) is equal to K_m times the voltage
9. The relationship between the level (h) and outlet flow rate (q_o) is linear, $q_o = h/R$ (R is the tank resistance in sec/m^2).

(i) Develop a differential equation to describe the relationship between the liquid level (h) and the inlet flow rate (q_i).

[2 marks]

(ii) Write the transfer function with assumption that the tank starts empty.

[2 marks]

(iii) Draw a block diagram for the whole closed-loop control system.

[4 marks]

(iv) Write the total feedback control system.

[4 marks]

SOLUTION

(i) Differential equation:

Based the mass conservation law, we have:

$$\rho\Delta V = \rho q_i - \rho q_o$$

Because density ρ is constant, we have:

$$A\frac{dh}{dt} = q_i - q_o$$

Substituting $q_o = h/R$, we have the differential equation:

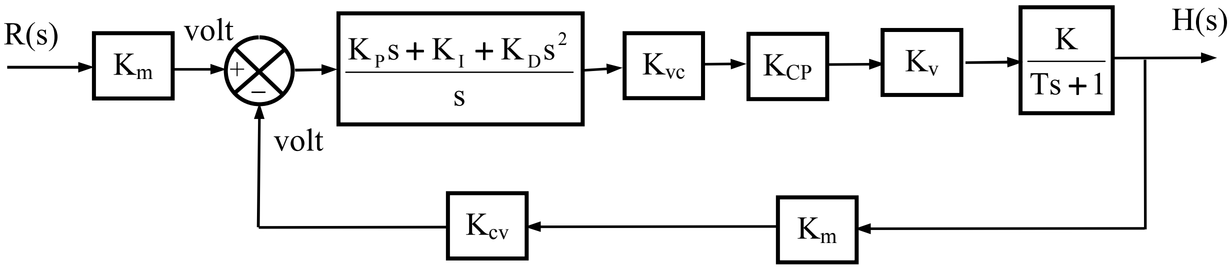
$$A\frac{dh}{dt} + \frac{h}{R} = q_i$$

(ii) Transfer function:

$$G(s) = \frac{H(s)}{Q_i(s)}\frac{K}{Ts+1}$$

where K = R (constant), T = AR (time constant).

(iii) Block diagram: Based on the above schematic diagram, the block diagram for the whole system is as follows:



(iv) Total feedback transfer function: Based on the above block diagram, the total feedback transfer function is:

$$F.B.T.F = \frac{H(s)}{R(s)} = K_m \frac{\frac{K_P s + K_I + K_D s^2}{s} K_{vc} K_{cp} K_v \frac{K}{Ts+1}}{1 + \frac{K_P s + K_I + K_D s^2}{s} K_{vc} K_{cp} K_v \frac{K}{Ts+1} K_m K_{cv}}$$

$$= \frac{K_m \left(K_P s + K_I + K_D s^2\right) K_{vc} K_{cp} K_v K}{s(Ts+1) + \left(K_P s + K_I + K_D s^2\right) K_{vc} K_{cp} K_v K K_m K_{cv}}$$

END OF CLASS TEST 2 – 36 MARKS AVAILABLE